of strength and the kinetic concept of strength. As in the case of a constant load, it can be assumed that Γ is independent of t — regardless of temperature, each soil corresponds to a single value of Γ (except for sand with t = −15°C, this result apparently being connected with a deviation from the pressure dependence of the freezing point of water described by Eq. (3)).

Although the form of the temperature-time (or rate) dependence of strength will be refined later, it can be taken as an established fact that temperature is connected with only one of the two parameters of the rheological curve. This makes it possible to significantly shorten and simplify tests and calculations performed to determine the strength of frozen soil.

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PROBLEM OF THE THEORY OF BEAMS WITH INITIAL STRESSES

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Aysymptotic averaging methods — which have been widely used for monolithic composites (see [1-3] and the accompanying bibiliographies) — are now being used to study bodies with a periodic structure that occupy thin regions: plates and beams [4-6]. In the present study, we make a transition from a three-dimensional problem of the theory of elasticity with initial stresses in the region of the small diameter ε (which is formalized in the form $\varepsilon \to 0$) to a problem of beam theory. In the general case, the new problem (which is asymptotically exact) differs from the classical problem. It coincides with the classical problem for uniform beams, however, i.e., the difference between the asymptotic and classical theories is seen for beams of complex structure. The use of such beams in modern structures makes the corrections introduced by asympototic theory practically important. The difference between the given problem and the problem examined in [6] is the asymmetry of the coefficients. This leads to the appearance of new elements in the use of asymptotic methods, as well as to several new cellular problems. As will be seen from the below discussion ε, the order of the initial stresses σ^*_{ij} relative to the diameter of the region ε plays a significant role in the problem. To account for this, we take the initial stresses in the form $\sigma^*_{ij} = \varepsilon^{-2}\sigma^{*(-2)}_{ij} + \varepsilon^{-1}\sigma^{*(-1)}_{ij} + \ldots$, corresponding to bending of the beam or its axial tension with fixed forces. The axial tension of a beam with fixed strains, when σ^*_{ij} is on the order of ε^{-4}, leads to results similar to [7-9] for monolithic bodies. This case is not examined here.

Formulation of the Problem. We will examine a body of periodic construction obtained by repeating a certain unit cell (UC) P_ε (ε is its characteristic dimension) along the Ox_1 axis (see Fig. 1). As a result, we have a body of periodic structure with the character-

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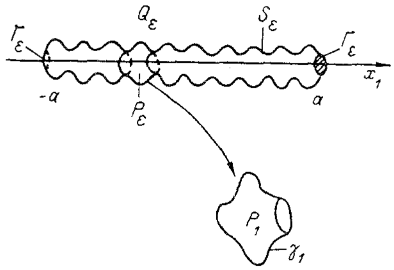

Fig. 1

istic diameter $\varepsilon \ll 1$ — a beam. The equations of equilibrium (vibration) of the beam as a three-dimensional elastic body with initial stresses will be written in the form [10]

$$\left(\mathscr{A}_{ijkl}(\mathbf{x}, \mathbf{x}/\varepsilon)\, u^{\varepsilon}_{k,l}\right)_{,j} = f_i \text{ in } Q_\varepsilon,$$
$$\mathscr{A}_{ijkl}(\mathbf{x}, \mathbf{x}/\varepsilon)\, u^{\varepsilon}_{k,l} n_j = 0 \text{ on } S_\varepsilon,\ \mathbf{u}^{\varepsilon}(\mathbf{x}) = 0 \text{ on } \Gamma_\varepsilon. \tag{1}$$

Here, $\partial/\partial x_j =, j$; Q_ε, S_ε, Γ_ε is the region occupied by the beam and its surface (see Fig. 1); $[-a, a]$ is the projection of the region Q_ε on the Ox_1 axis; $\mathbf{u}^\varepsilon$ are the displacements; $\mathbf{f} = (\varepsilon^{-2}f^0{}_1, \varepsilon^{-2}f^0{}_2, \varepsilon^{-2}f^0{}_3)$ are the body forces (for the problem of natural vibration $\mathbf{f} = \varepsilon^{-2}\rho(\mathbf{x}/\varepsilon)\omega\mathbf{u}^\varepsilon$), where ω is the natural frequency); The multiplier ε^{-2} is introduced so that when $\varepsilon \to 0$ the limiting values are nontrivial [6]; $\mathscr{A}_{ijk\ell}(\mathbf{x}, \mathbf{x}/\varepsilon)$ are known [10] combinations of the tensor of elastic constants $\varepsilon^{-4}a_{ijk\ell}(\mathbf{x}/\varepsilon)$ and the initial stresses $\sigma^*{}_{ij}(\mathbf{x}, \mathbf{x}/\varepsilon)$:

$$\mathscr{A}_{ijkl}(\mathbf{x}, \mathbf{x}/\varepsilon) = \varepsilon^{-4}a_{ijkl}(\mathbf{x}/\varepsilon) + \varepsilon^{-2}\mathscr{A}^{(-2)}_{ijkl}(\mathbf{x}, \mathbf{x}/\varepsilon) + \varepsilon^{-1}\mathscr{A}^{(-1)}_{ijkl}(\mathbf{x}, \mathbf{x}/\varepsilon) + \ldots, \tag{2}$$

where $\mathscr{A}^{(m)}_{ijkl}(\mathbf{x}, \mathbf{x}/\varepsilon) = \sigma^{*(m)}_{jl}(\mathbf{x}, \mathbf{x}/\varepsilon)\,\delta_{ik}$ $(m = -2, -1, \ldots)$;

δ_{ij} is the Kronecker delta.

Note 1. In connection with the fact that the coefficients (2) differ from the coefficients normally used (which are on the order of unity for monolithic bodies [1-3, 7, 8], on the order of ε^{-3} for plates [4, 5], and on the order of ε^{-4} for beams [6]), we will comment briefly on the terms in (2). The term $\varepsilon^{-4}a_{ijk\ell}$ guarantees that the bending stiffness of the beam will be nontrivial at $\varepsilon \to 0$ (as is known, the stiffness of a beam in bending is proportional to its diameter to the fourth power [11]). The term $\varepsilon^{-2}\sigma^*{}_{j\ell}{}^{(-2)}$ corresponds to tension of the beam by a force which is independent of ε (constant tension of the beam). In fact, multiplied by the cross-sectional area — which is on the order of ε^2 — this quantity remains independent of ε. The term $\varepsilon^{-1}\sigma^*{}_{j\ell}{}^{(-1)}$ corresponds to the stresses which arise in the beam during bending, as follows from [6].

The functions $\rho(\mathbf{y})$, $a_{ijk\ell}(\mathbf{y})$, $\sigma^*{}_{j\ell}{}^{(m)}(x_1, \mathbf{y})$ are periodic with respect to y_1 and have the period $[0, m]$, where $[0, m]$ is the projection of the UC $P_1 = \varepsilon^{-1}P_\varepsilon = \{\mathbf{y} = \varepsilon^{-1}\mathbf{x}: \mathbf{x} \in P_\varepsilon\}$ on the Oy_1 axis.

Asymptotic Expansion. Let us examine problem (1-2) when $\varepsilon \to 0$. To do this, we make use of the asymptotic expansion [6]

$$\mathbf{u}^\varepsilon = \mathbf{u}^{(0)}(x_1) + \varepsilon\mathbf{u}^{(1)}(x_1, \mathbf{y}) + \ldots = \sum_{k=0}^{\infty} \varepsilon^k \mathbf{u}^{(k)}(x_1, \mathbf{y}),\ \langle \mathbf{u}^{(k)} \rangle = 0 \text{ at } k \geqslant 1, \tag{3}$$
$$\sigma_{ij} = \varepsilon^{-4}\sigma^{(-4)}_{ij}(x_1, \mathbf{y}) + \varepsilon^{-3}\sigma^{(-3)}_{ij}(x_1, \mathbf{y}) + \ldots = \sum_{m=-4}^{\infty} \varepsilon^m \sigma^{(m)}_{ij}(x_1, \mathbf{y}).$$

Here, x_1 is a slow variable along the axis of the beam $[-a, a]$; $\mathbf{y} = \mathbf{x}/\varepsilon$ is a fast variable; the functions in the right sides are assumed to be periodic with respect to y_1, with the UC $[0, m]$; $\langle\cdot\rangle = (1/m)\int_{P_1} \cdot\, d\mathbf{y}$ is the mean value of P_1 over the UC.

Analysis of problem (1)-(2) for $\varepsilon \to 0$ breaks down into two stages [4]. The first entails obtaining the equations of equilibrium. As follows from [6], it is not involved with local governing equations (in the given case, with the coefficients $\mathscr{A}_{ijk\ell}$), and it is the only stage for any equation of state. There are the following equations for the forces $N^{(m)}{}_{ij} = \langle\sigma^{(m)}{}_{ij}\rangle$, bending moments $M^{(m)}_{\alpha j} = \langle y_\alpha \sigma_{j1}{}^{(m)}\rangle$, and turning moment $\mathscr{M} = M^{(-3)}_{32} - M^{(-3)}_{23}$:

$$N^{(m)}_{\alpha 1,1x} = F_\alpha \ (F_\alpha = 0 \text{ at } m = -3, \ F_\alpha = \langle f^0_\alpha \rangle \text{ at } m = -2); \tag{4a}$$

$$-M^{(-3)}_{\beta;1x} + N^{(-2)}_{1\beta} = \langle f^0_1 y_\beta \rangle \text{ (in the problem of natural vibration } f^0_\beta = \langle \rho \rangle \omega u^{(0)}_\beta); \tag{4b}$$

$$\mathcal{M}_{,1x} + (N^{(-2)}_{32} - N^{(-2)}_{23}) = \langle f^0_2 y_3 \rangle - \langle f^0_3 y_2 \rangle, \tag{4c}$$

while the following relations [6] are satisfied for local stresses $\sigma^{(m)}{}_{ij}$

$$\sigma^{(m)}_{ij,jy} = 0 \text{ in } Q^\varepsilon_1, \ \sigma^{(m)}_{ij} n_j = 0 \text{ on } \gamma^\varepsilon_1, \tag{5}$$

where $Q_1{}^\varepsilon = \{(x_1, y_2, y_3): \ \mathbf{y} = \mathbf{x}/\varepsilon, \ \mathbf{x} \in Q_\varepsilon\}$; $\gamma_1{}^\varepsilon$ is the lateral (free) surface $Q_1{}^\varepsilon$. Relations (4a)-(4c), (5) are independent of the governing relations. Here and below, the Latin indices take the values 1, 2, and 3 and the Greek indices take the values 2 and 3; $\partial/\partial x_1$ = 1x; $\partial/\partial y_i$ = ,iy.

Note 2. By the additional stresses [10] σ_{ij}, here and below we mean the quantities

$$\sigma_{ij} = \mathcal{A}_{ijkl} u^\varepsilon_{k,l} = (\varepsilon^{-4} a_{ijkl} + \varepsilon^{-2} \mathcal{A}^{(-2)}_{ijkl} + \varepsilon^{-1} \mathcal{A}^{(-1)}_{ijkl} + \ldots) u^\varepsilon_{k,l}. \tag{6}$$

The total stresses are equal to the sum of the initial and additional stresses: $\sigma^*_{ij} + \sigma_{ij}$.

The second stage of analysis of the problem consists of obtaining the governing relations for the beam, which establish the relationship between $N^{(m)}{}_{\alpha_1}$, $M^{(-3)}{}_\beta$, $\mathcal{M}$ and the strain characteristics. In contrast to the first stage, this stage does involve local governing relations (with initial stresses, in our case) and is the main stage in the present investigation.

Note 3. With the use of two-scale expansion, the differentiation operators are represented in the form of the sum of the operators $\partial/\partial x_i$ and $\partial/\partial y_i$. For the functions of the arguments x_1 and $\mathbf{y} = (y_1, y_2, y_3)$, in the right sides of (3), this representation takes the form [6]

$$\varepsilon^{-1}\partial/\partial y_\alpha \ (\alpha = 2, 3), \ \varepsilon^{-1}\partial/\partial y_1 + \partial/\partial x_1.$$

Having inserted (3) into (6), with allowance for note 2 we obtain

$$\sum_{m=-4}^{\infty} \varepsilon^m \sigma^{(m)}_{ij} = \sum_{k=0}^{\infty} \varepsilon^k (\varepsilon^{-4} a_{ijkl} + \varepsilon^{-2} \mathcal{A}^{(-2)}_{ijkl} + \varepsilon^{-1} \mathcal{A}^{(-1)}_{ijkl} + \ldots) (u^{(k)}_{k,1x} + \varepsilon^{-1} u^{(k)}_{k,ly} + \ldots). \tag{7}$$

Equating the terms with identical powers of ε in (6), we have

$$\sigma^{(m)}_{ij} = a_{ijk1} u^{(m+4)}_{k,1x} + a_{ijkl} u^{(m+5)}_{k,ly} \ (m = -4, -3); \tag{8a}$$

$$\sigma^{(-2)}_{ij} = a_{ijk1} u^{(2)}_{k,1x} + \mathcal{A}^{(-2)}_{ijk1} u^{(0)}_{k,1x} + a_{ijkl} u^{(3)}_{k,ly} + \mathcal{A}^{(-2)}_{ijkl} u^{(1)}_{k,ly}, \tag{8b}$$

etc. We will examine problem (5), (8) when m = −4, with the following conditions from (3): $\mathbf{u}^{(1)}$ is periodic with respect to y_1 and has the UC [0, m] and $\langle \mathbf{u}^{(1)} \rangle = 0$. Allowing for the fact that the function of the argument x_1 plays the role of a parameter, its solution can be found in the form [1, 2, 4]

$$\mathbf{u}^{(1)} = -y_\alpha u^{(0)}_{\alpha,1x}(x_1)\mathbf{e}_1 + \mathbf{U}(\mathbf{y})\varphi(x_1) + \mathbf{V}(x_1) \tag{9}$$

($\{\mathbf{e}_i\}$ are basis vectors of the coordinate system). In obtaining (9), we make use of the functions $\mathbf{X}^{p\alpha(\nu)}$ (ν = 0, 1) – the solutions of the so-called celluar problem (CP) of beam theory introduced in [6]:

$$\begin{aligned} &(a_{ijkl}(\mathbf{y}) X^{p\alpha(\nu)}_{k,ly} + a_{ijp1}(\mathbf{y}) y^\nu_\alpha)_{,jy} = 0 \text{ in } P_1, \\ &(a_{ijkl}(\mathbf{y}) X^{p\alpha(\nu)}_{k,ly} + a_{ijp1}(\mathbf{y}) y^\nu_\alpha) n_j = 0 \text{ on } \gamma_1, \end{aligned} \tag{10}$$

$\mathbf{X}^{p\alpha(\nu)}(\mathbf{y})$ is periodic with respect to y_1 and has the period [0, m] and $\langle \mathbf{X}^{p\alpha(\nu)} \rangle = 0$ (γ_1 is the lateral (free surface of P_1 (see Fig. 1). Here, we considered that [6]

$$\mathbf{X}^{\beta 1(0)}(\mathbf{y}) = -y_\beta \mathbf{e}_1 \text{ and } u_1^{(0)}(x_1) = 0.$$

The function $\mathbf{U}(\mathbf{y}) = y_\beta s_{\tilde\beta} \mathbf{e}_\beta$, where $s_1 = 0$, $s_2 = 1$, $s_3 = -1$, $\tilde\beta = \begin{cases} 3 \text{ at } \beta = 2 \\ 2 \text{ at } \beta = 3 \end{cases}$ is connected with the torsion of a rod (this subject is examined in more detail in [6]).

In accordance with [6], substitution of (10) into (8a)-(8b) gives the equalities

$$\sigma_{ij}^{(-4)} = 0; \tag{11a}$$

$$\sigma_{ij}^{(-3)} = a_{ijkl}(\mathbf{y}) u_{k,ly}^{(2)} + a_{ij11}(\mathbf{y}) y_\alpha u_{\alpha,1x1x}^{(0)}(x_1) + a_{ij11}(\mathbf{y}) V_{1,1x}(x_1) + a_{ij\beta 1}(\mathbf{y}) s_{\tilde\beta} y_\beta \varphi_{,1x}(x_1). \tag{11b}$$

Let us examine problem (5) with the following conditions when m = −3 (taking into account (11a-11b): $\mathbf{u}^{(2)}$ is periodic with respect to y_1 and has the UC [0, m] and $\langle \mathbf{u}^{(2)} \rangle = 0$. Its solution can be written in the form [6]

$$\mathbf{u}^{(2)} = \mathbf{X}^{11(0)}(\mathbf{y}) V_{1,1x}(x_1) - y_\alpha V_{\alpha,1x}(x_1) \mathbf{e}_1 + \mathbf{X}^{1\beta(1)}(\mathbf{y}) u_{\beta,1x1x}^{(0)}(x_1) + \mathbf{X}^{(3)}(\mathbf{y}) \varphi_{,1x}(x_1) + y_\beta s_{\tilde\beta} \mathbf{e}_\beta \psi(x_1) + \mathbf{W}(x_1), \tag{12}$$

where $\mathbf{X}^{(3)}(\mathbf{y})$ is the solution of the CP describing torsion [6]:

$$\left(a_{ijkl}(\mathbf{y}) X_{k,ly}^{(3)} + a_{ij\beta 1}(\mathbf{y}) s_\beta y_{\tilde\beta}\right)_{,jy} = 0 \text{ in } P_1,$$

$$\left(a_{ijkl}(\mathbf{y}) X_{k,ly}^{(3)} + a_{ij\beta 1}(\mathbf{y}) s_\beta y_{\tilde\beta}\right) n_j = 0 \text{ on } \gamma_1,$$

$\mathbf{X}^{(3)}(\mathbf{y})$ is periodic with respect to y_1 and has the period [0, m] and $\langle \mathbf{X}^{(3)} \rangle = 0$.

After averaging P_1 over the UC, insertion of (12) into (8a) with m = −3 gives the following (see [6] for more detail)

$$\begin{aligned} N_{11}^{(-3)} &= A_{111}^0 V_{1,1x} + A_{11\alpha}^1 u_{\alpha,1x1x}^{(0)} + B_{11}^0 \varphi_{,1x}, \\ M_{\beta 1}^{(-3)} &= A_{\beta 11}^1 V_{1,1x} + A_{\beta 1\alpha}^2 u_{\alpha,1x1x}^{(0)} + B_{\beta 1}^1 \varphi_{,1x}, \\ \mathscr{M} &= A_{001}^1 V_{1,1x} + A_{00\alpha}^2 u_{\alpha,1x1x}^{(0)} + B_{00}^1 \varphi_{,1x}. \end{aligned} \tag{13}$$

Here, the coefficients $A_{ijk}^{\mu+\nu}$, $B^\mu{}_{ij}$ (μ, ν = 0.1) are given by Eqs. (3.30) from [6] and are expressed through the functions entering into the CP. This means in particular that the total stresses and moments in the beam will be $N^*_{11} + N_{11}^{(-3)}$, $M^*_{\beta 1} + M_{\beta 1}^{(-3)}$, $\mathscr{M}^* + \mathscr{M}$, where the asterisks denote the forces and moments corresponding to the initial stresses σ^*_{ij}. These quantities are calculated from standard formulas [11]. As can be seen, for thin beams, torsional rigidity is independent of the initial stresses − in contrast to [10].

The authors of [6] did not further investigate the governing relations for the case of the absence of initial stresses − when, due to $\sigma^*{}_{ij}^{(-2)} = \sigma^*{}_{ji}^{(-2)}$, we have $N^{(-2)}{}_{ij} = N^{(-2)}{}_{ji}$ because the above-indicated symmetry made it possible to exclude the quantities; $N_{ij}^{(-2)}$ from (4a)-(4b) without any additional information on them. In our case $N_{ij}^{(-2)}$ does not have the indicated symmetry, and obtaining the limiting problem requires further study of asymptote (3). Specifically, we need to examine $N_{ij}^{(-2)}$.

Let us insert $\mathbf{u}^{(1)}$ into (8b) in accordance with (10). Then considering that $u_1^{(0)}(x_1) = 0$, we obtain

$$\sigma_{ij}^{(-2)} = a_{ijk1} u_{k,1x}^{(2)} + a_{ijkl} u_{k,ly}^{(3)} + \mathscr{A}_{ijk1}^{(-2)} u_{k,1x}^{(0)} - \mathscr{A}_{ij1\alpha}^{(-2)} u_{\alpha,1x}^{(0)} + \mathscr{A}_{ij\beta\tilde\beta}^{(-2)} s_{\tilde\beta} \varphi(x_1) = f_{ij} + \left[\left(\sigma_{j1}^{*(-2)} \delta_{i\alpha} - \sigma_{j\alpha}^{*(-2)} \delta_{i1}\right) u_{\alpha,1x}^{(0)} + \sigma_{j\tilde\beta}^{*(-2)} \delta_{i\beta} s_{\tilde\beta} \varphi(x_1)\right]. \tag{14}$$

Here, f_{ij} represents the first two terms in the right side of (14). We find that $f_{ij} = f_{ji}$ for these terms, by virtue of the symmetry of the elastic constants ($a_{ijk\ell} = a_{ijk\ell}$). Terms which are asymmetric with respect to i and j enter into the right side of (14) only in the expression in square brackets. Then the forces $N_{ij}^{(-2)}$ can be represented in the form

$$N_{ij}^{(-2)} = \langle f_{ij} \rangle + \left\langle \sigma_{j1}^{*(-2)} \delta_{i\alpha} - \sigma_{j\alpha}^{*(-2)} \delta_{i1} \right\rangle u_{\alpha,1x}^{(0)}(x_1) + \left\langle \sigma_{j\tilde\beta}^{*(-2)} \right\rangle \delta_{i\beta} s_{\tilde\beta} \varphi(x_1). \tag{15}$$

Tension, Bending. The below relation follows in particular from (15)

$$N_{1\beta}^{(-2)} = N_{\beta 1}^{(-2)} + K_\beta, \tag{16}$$

where

$$K_\beta = \langle -\sigma_{11}^{*(-2)}\delta_{\beta\alpha} + \sigma_{\beta 1}^{*(-2)}\delta_{\beta\alpha} - \sigma_{\beta\alpha}^{*(-2)}\rangle u_{\alpha,1x}^{(0)}(x_1) + \langle \sigma_{12}^{*(-2)} - \sigma_{13}^{*(-2)}\rangle \varphi(x_1). \tag{17}$$

With allowance for (16), we proceed as follows to exclude $N_{1\beta}^{(-2)}$ from (4a)-(4b): differentiating (4b) and using (16), we obtain

$$0 = (-M_{\beta 1,1x}^{(-3)} + N_{1\beta}^{(-2)})_{,1x} = (-M_{\beta 1,1x}^{(-3)} + N_{\beta 1}^{(-2)} + K_\beta)_{,1x} =$$
$$= -M_{\beta 1,1x}^{(-3)} + N_{\beta 1,1x}^{(-2)} + K_{\beta,1x} = -M_{\beta 1,1x}^{(-3)} - \langle f_\beta^0\rangle + K_{\beta,1x}.$$

From this, we obtain an equation which replaces (4a) and (4b) when m = −2:

$$-M_{\beta 1,1x}^{(-3)} = \langle f_\beta^0\rangle - K_{\beta,1x}. \tag{18}$$

Torsion. The governing equations for the turning moment $\mathscr{M} = M_{23}^{(-2)} - M_{32}^{(-2)}$ are contained in (13). As regards the equilibrium equation, we can use (4b) to obtain

$$-\mathscr{M}_{,1x} + (N_{32}^{(-2)} - N_{23}^{(-2)}) = \langle f_2^0 y_3\rangle - \langle f_3^0 y_2\rangle.$$

Here, as above, we encounter a situation connected with the asymmetry of $N_{ij}^{(-2)}$. In accordance with (16), we have

$$N_{23}^{(-2)} - N_{32}^{(-2)} = K \equiv \langle \sigma_{31}^{*(-2)}\delta_{2\alpha} - \sigma_{21}^{*(-2)}\delta_{3\alpha}\rangle u_{\alpha,1x}^{(0)} + \langle \sigma_{22}^{*(-2)} + \sigma_{33}^{*(-2)}\rangle\varphi. \tag{19}$$

In sum

$$-\mathscr{M}_{,1x} - K = \langle f_2^0 y_3\rangle - \langle f_3^0 y_2\rangle. \tag{20}$$

Complete System of Equations (limiting problem). When m = −2, Eqs. ((4a), (18), and (20) and the boundary conditions (obtained in [6])

$$u_\beta^{(0)}(\pm a) = 0,\quad u_{\beta,1x}^{(0)}(\pm a) = 0,\quad V_1(\pm a) = 0,\quad \varphi(\pm a) = 0\ (\beta = 2,3)$$

constitute a complete system of equations and boundary conditions for determination of the functions $u_2^{(0)}$, $u_3^{(0)}$, V_1, ϕ.

Cylindrical Rod of a Uniform Isotropic Material. Let $Q_\varepsilon = [-a, a] \times S_\varepsilon$ be a cylinder, and let us assume that the material of the rod is uniform and istropic. The initial stresses are determined from the solution of the problem of the theory of elasticity for assigned forces N_{11}. Its solution is $\sigma^*_{ij} = \varepsilon^{-2}N_{11}\delta_{i1}\delta_{j1}$ (this can easily be proven). Then (17) and (19) take the form $K_\beta = -N_{11}\, u_{\beta,1x}^{(0)}$ and K = 0, respectively. As a result, (18) and (20) become the classical equations of the bending of beam with initial stresses [11].

Note on the Behavior of the Formulas for K_β, K. Terms of the form $\langle\sigma^*{}_{ij}^{(-2)}\rangle$ enter into Eqs. (17) and (19) for K_β, K. These terms can be calculated on the basis of the theory of elasticity, which describes the preliminary tension of the rod. Let the initial stresses be determined from the solution of the theory of elasticity problem

$$\sigma_{ij,j}^* = \varepsilon^a f_i \text{ in } Q_\varepsilon,$$
$$\sigma_{ij}^* n_j = \varepsilon^b g_i \text{ on } S_\varepsilon,\quad \sigma_{ij}^* n_j = h \text{ on } \Gamma_\varepsilon. \tag{21}$$

Shown below are relations which follow in particular from insertion of the asymptotic expansion for the initial stresses $\sigma^*_{ij} = \varepsilon^{-4}\sigma^*{}_{ij}^{(-4)} + -\varepsilon^{-3}\sigma^*{}_{ij}^{(-3)} + \varepsilon^{-2}\sigma^*{}_{ij}^{(-2)} + \ldots$ into (21):

$$\sigma_{ij,jy}^{*(-2)} = F_i \text{ in } Q_1^\varepsilon\quad (F_i = 0 \text{ at } a \neq -3,\ F_i = f_i \text{ at } a = -3),$$
$$\sigma_{ij}^{*(-2)} n_j = G_i \text{ on } \gamma_1^\varepsilon\quad (G_i = 0 \text{ at } b \neq -2,\ G_i = g_i \text{ at } b = -2),$$
$$\sigma_{ij}^{*(-2)}(x_1, y) \text{ is periodic with respect to } y_1 \text{ and has the period } [0, m]. \tag{22}$$

We multiply the first equation in (22) by y_α and integrate over the UC P_1 with allowance for the remaining conditions from (22). We obtain

$$\langle\sigma_{i\alpha}^{*(-2)}\rangle = -\langle F_i y_\alpha\rangle - \langle G_i y_\alpha\rangle, \quad i = 1, 2, 3, \quad \alpha = 2, 3. \tag{23}$$

The last expression is equal to zero when $a \neq -3$, $b \neq -2$. It follows from Eq. (23) that if the initial stresses are not due to body or surface forces of the indicated magnitude, then $K_\beta = -\langle\sigma^*_{11}{}^{(-2)}\rangle u_{\beta,1x}^{(0)}$, $K = 0$, i.e., in the given case the asymptotic formulas coincide with the classical formulas [11]. When $a = -3$ or $b = -2$, the expressions for K_β and K may include nontrivial terms containing $\langle\sigma^*_{\alpha\alpha}{}^{(-2)}\rangle$ (connected with transverse compression of the rod) or $\langle\sigma^*_{\alpha 1}{}^{(-2)}\rangle$ (connected with transverse shear strains).

Stability of the Beam in Tension. In the case of pure torsion (in the absence of axial tension and normal deflections and when $f_2^0 = f_3^0 = 0$), then torsion equations take the form

$$B_{00}^1\varphi_{,1x1x} + \langle\sigma_{22}^{*(-2)} + \sigma_{33}^{*(-2)}\rangle\varphi = 0 \tag{24}$$

with the boundary conditions $\phi(\pm a) = 0$. The resulting problem is not the same as the problem presented in [10, Sec. 6.4] on the torsion of a pretensioned rod. In particular, when $a \neq -3$ and $b \neq -2$, Eq. (24) takes the form $B^1_{00}\phi_{,1x1x} = 0$. The latter means that the structure remains stable during torsion.

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